New Insights into Lower Bound for Lie Groups and their Applications

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***Abstract*—This article presents a comprehensive review of recent advances in intrinsic Cramér-Rao bounds (ICRBs) for Lie groups (LGs), which play a pivotal role in addressing estimation problems involving parameters and/or observations constrained by geometric structures. The review encompasses both deterministic and Bayesian frameworks, with a detailed examination of their formulation, derivation, and theoretical foundations. Furthermore, we underscore significant theoretical contributions and extend the discussion to practical estimation challenges, offering insights into their applicability. Emphasis is placed on methodologies for validating these bounds, providing a robust framework for performance evaluation across a variety of estimation problems in engineering and applied sciences.**

***Index Terms*—Intrinsic Cramér-Rao Bounds, Lie Groups.**

I. Introduction

The Cramér-Rao bound (CRB) provides a lower limit on the precision achievable by any unbiased estimator $\widehat{\mathbf{x}}$ of a vector $\mathbf{x}$ that parametrizes a family of p.d.f.s $p(\mathbf{z}|\mathbf{x})$, from a sample of observations $\mathbf{z}_1, \ldots, \mathbf{z}_n$. This bound plays a pivotal role in evaluating the performance of estimation methods within Euclidean spaces. However, in many practical problems, the observed data or underlying hidden variables are subject to nonlinear geometric restrictions, which can be modeled by constraining parts of the model to manifolds [1], [2]. For example, problems with parameters in Lie Groups (LGs), -a smooth manifold with group structure-, appear in numerous applications across signal processing [3], robotics [4], [5], and computer vision [5]. Example of notable LGs are the *Special Orthogonal Group* $SO(3)$ of rotation matrices in a 3-D space, the *Special Euclidean Group* $SE(3)$, which represents rigid body motions in 3-D space, combining rotations and translations. For instance, registration problems often involve transformations between two unaligned images that belong to the *special Euclidean group* $SE(3)$ or the *similarity group* $Sim(3)$ [6].

In line with the widespread of estimation problems involving LGs, it becomes essential to establish geometry-aware algorithms and lower bounds on intrinsic error measures to assess their performance. To address the latter, various intrinsic Cramér-Rao bounds (ICRBs) have been developed, encompassing both Bayesian and deterministic (i.e. non-Bayesian) estimation paradigms. This has led to the introduction of deterministic LG-CRBs [7]–[10] and Bayesian LG-CRBs [11]–[13].

This article provides a detailed review of recent advances in lower bounds for Lie groups (LGs), with a particular emphasis on the authors' contributions to both deterministic estimation problems and Bayesian frameworks. The discussion underscores key theoretical developments and explores their application to practical estimation challenges. The paper is structured as follows: Section II introduces general concepts related to LGs. Section III presents the intrinsic LG-Cramér-Rao bound (LG-CRB) for deterministic parameter estimation and derives the Slepian-Bangs formula counterpart for LGs. Section IV focuses on the intrinsic Bayesian bound for LGs. To demonstrate the utility of these bounds, Section V validates the derived closed-form expressions for both LG-CRB and LG-BCRB, through numerical simulations, on two practical estimation problems: Wahba's problem on $SO(3)$ and a pose estimation problem on $SE(3)$.

II. Background on Lie Groups

A. Lie Group Definitions

A matrix Lie Group (LG), $G \subset \mathbb{R}^{n\times n}$, is a matrix space that satisfies the properties of both a smooth manifold and a group. This structure defines a tangent space, known as the Lie algebra $\mathfrak{g}$. The *exponential map*, $\mathrm{Exp}_G : \mathfrak{g} \to G$, and the *logarithmic map*, $\mathrm{Log}_G : G \to \mathfrak{g}$, establish a relationship between elements of the LG and its Lie algebra.

Since the Lie algebra $\mathfrak{g}$ is isomorphic to $\mathbb{R}^m$, two bijections can be defined: $[.]^\wedge : \mathbb{R}^m \to \mathfrak{g}$ and $[.]^\vee : \mathfrak{g} \to \mathbb{R}^m$. Using these, the exponential and logarithmic mappings can be expressed as:

$$\forall\, \mathbf{a} \in \mathbb{R}^m, \quad \mathrm{Exp}_G^\wedge(\mathbf{a}) = \mathrm{Exp}_G\left([\mathbf{a}]_G^\wedge\right), \tag{1}$$

$$\forall\, \mathbf{X} \in G, \quad [\mathrm{Log}_G(\mathbf{X})]_G^\vee = \mathrm{Log}_G^\vee(\mathbf{X}). \tag{2}$$

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For example,

- The Special Orthogonal group $SO(3)$ is a Lie group of $3D$ rotation matrices $\mathbf{R}$, such that $\mathbf{R}^\top\mathbf{R} = \mathbf{I}$ and $|\mathbf{R}| = 1$, where $^\top$ denotes the transpose operator and $|.|$ the matrix determinant. $SO(3)$ describes all possible rotations of a physical object in 3D space and its Lie algebra $\mathfrak{so}(3)$ correspond to the set of skew-symmetric matrices. More precisely, $\mathfrak{so}(3) = \{[\mathbf{w}]_\times | \mathbf{w} \in \mathbb{R}^3\}$ where $[.]_\times$ denotes the operator which transforms a vector to a skew-symmetric matrix. Consider $\mathbf{R} \in SO(3)$ such as $\mathbf{R} = \mathrm{Exp}^\wedge_{SO(3)}(\mathbf{w})$. The exponential and logarithmic applications are given by the Rodrigues rotation formula,

$$\mathrm{Exp}^\wedge_{SO(3)}(\mathbf{w}) = \mathbf{I}_3 + \frac{[\mathbf{w}]_\times}{\|\mathbf{w}\|}\sin(\|\mathbf{w}\|) + \frac{[\mathbf{w}]^2_\times}{\|\mathbf{w}\|^2}(1-\cos(\|\mathbf{w}\|)) \tag{3}$$

$$\mathrm{Log}^\vee_{SO(3)}(\mathbf{R}) = \frac{\|\mathbf{w}\| \left[\mathbf{R}-\mathbf{R}^\top\right]^\vee}{2\sin(\|\mathbf{w}\|)} \tag{4}$$

- $SE(3)$ defines the semi-direct product group between $SO(3)$ and $\mathbb{R}^3$. From an application point of view, it can be used to model the pose of a camera or a robot $SE(3) = \left\{\mathbf{X}_0 = \begin{bmatrix}\mathbf{R} & \mathbf{p}\\ \mathbf{0} & 1\end{bmatrix} | \mathbf{R} \in SO(3), \mathbf{p} \in \mathbb{R}^3\right\}$, and its Lie algebra has the following structure $\mathfrak{se}(3) = \left\{\mathbf{A} = \begin{bmatrix}[\mathbf{w}] & \mathbf{u}\\ \mathbf{0} & 0\end{bmatrix} | \mathbf{w} \in \mathbb{R}^3, \mathbf{u} \in \mathbb{R}^3\right\}$. The exponential and logarithm operators can be built from $\mathrm{Log}^\vee_{SO(3)}(.)$ and $\mathrm{Exp}^\wedge_{SO(3)}(.)$. Indeed, if $\mathbf{X}_0 \in SE(3)$:

$$\mathbf{X}_0 = \begin{bmatrix}\mathbf{R} & \mathbf{p}\\ \mathbf{0} & 1\end{bmatrix} = \mathrm{Exp}^\wedge_{SE(3)}(\mathbf{w}), \ \mathbf{w} \in \mathbb{R}^6 \tag{5}$$

with $\mathbf{w} = \left[\mathbf{w}_R^\top, \mathbf{w}_p^\top\right]$, $,\mathbf{w}_R$ and $\mathbf{w}_p \in \mathbb{R}^3$, then:

$$\mathrm{Log}^\vee_{SE(3)}(\mathbf{X}_0) = \begin{bmatrix}\mathbf{D}(\mathbf{w}_R)^{-1}\,\mathbf{p}\\ \mathrm{Log}^\vee_{SO(3)}(\mathbf{R})\end{bmatrix}, \tag{6}$$

and for the exponential mapping:

$$\mathrm{Exp}^\wedge_{SE(3)}(\mathbf{w}) = \begin{bmatrix}\mathrm{Exp}^\wedge_{SO(3)}(\mathbf{w}_R) & \mathbf{M}(\mathbf{w}_R)\,\mathbf{w}_p\\ 0 & 1\end{bmatrix}, \tag{7}$$

where $\mathbf{M}(\mathbf{w}_R)$ and $\mathbf{D}(\mathbf{w}_R)$ are defined in [10].

B. The Baker-Campbell-Hausdorff Formula

The Baker-Campbell-Hausdorff (BCH) formula [14, Theorem 5.5] provides an explicit expression for the composition of exponential maps:

$$\mathrm{Log}^\vee_G\left(\mathrm{Exp}^\wedge_G(\mathbf{a})\,\mathrm{Exp}^\wedge_G(\mathbf{b})\right) = \mathrm{BCH}(\mathbf{a},\mathbf{b}),$$

which generally satisfies $\mathrm{BCH}(\mathbf{a},\mathbf{b}) \neq \mathbf{a}+\mathbf{b}$ due to the non-commutativity of most Lie groups. An approximation of BCH is given by:

$$\mathrm{Log}^\vee_G\left(\mathrm{Exp}^\wedge_G(\mathbf{a})\,\mathrm{Exp}^\wedge_G(\mathbf{b})\right) = \mathbf{b} + \psi_G(\mathbf{b})\boldsymbol{a} + \mathcal{O}(\|\mathbf{a}\|^2), \tag{8}$$

where $\psi_G(\mathbf{b}) \overset{def}{=} \sum_{n=1}^{+\infty} \frac{\mathrm{ad}_G(\mathbf{b})^n}{n!}$ is the inverse of the left Jacobian of G, and $\mathrm{ad}_G(\cdot) : \mathbb{R}^m \to \mathbb{R}^{m\times m}$ is the adjoint representation of $\boldsymbol{b}$ on $\mathfrak{g}$.

C. Estimation on Lie Groups

Let $\mathbf{Z} \in G' \subset \mathbb{R}^{n\times n}$ be observations from a matrix Lie group G'. These observations are connected to an unknown parameter $\mathbf{X}_0 \in G$ via the likelihood function $p(\mathbf{Z}|\mathbf{X}_0)$.

To quantify the discrepancy between $\widehat{\mathbf{X}}_0$ and X_0, an intrinsic metric is commonly used:

$$\boldsymbol{l}_G\left(\mathbf{X}_0, \widehat{\mathbf{X}}_0\right) \overset{def}{=} \mathrm{Log}^\vee_{G''}\left(\mathbf{X}_0^{-1}\widehat{\mathbf{X}}_0\right). \tag{9}$$

This metric leads to two key quantities: the intrinsic bias given by

$$\mathbf{b}_{\mathbf{Z}|\mathbf{X}_0} \overset{def}{=} \mathbb{E}_{p(\mathbf{Z}|\mathbf{X}_0)}\left(\boldsymbol{l}_G\left(\mathbf{X}_0, \widehat{\mathbf{X}}_0\right)\right), \tag{10}$$

and the Intrinsic Mean Squared Error (LG-MSE):

$$\mathbf{C}_{\mathbf{Z}|\mathbf{X}_0} \overset{def}{=} \mathbb{E}_{p(\mathbf{Z}|\mathbf{X}_0)}\left(\boldsymbol{l}_G\left(\mathbf{X}_0, \widehat{\mathbf{X}}_0\right)\boldsymbol{l}_G\left(\mathbf{X}_0, \widehat{\mathbf{X}}_0\right)^\top\right) \tag{11}$$

It is important to emphasize here that the defined expectations are according to a group measure. Traditionally, a Haar measure is used due to its invariance properties.

III. Deterministic Cramér-Rao Bounds on LGs

Throughout the entire section, $\mathbf{Z} = \{\mathbf{Z}_1, \ldots, \mathbf{Z}_N\}$ and $\mathbf{z} = \{\mathbf{z}_1, \ldots, \mathbf{z}_N\}$ denote sets of independent observations on G' and $\mathbb{R}^p$, respectively. The unknown LG parameter $\mathbf{X}_0$ on G, with dimension m, is characterized by the likelihood function $p(\mathbf{Z}|\mathbf{X}_0)$ or $p(\mathbf{z}|\mathbf{X}_0)$.

A. LG-CRB based on the Barankin Bound on LGs

To define a lower bound, a fundamental property in the Euclidean case is the strict-sense/uniform unbiasedness. This property is well-known in the Euclidean framework and allows to theorize the Barankin Bound (BB) [15]. An intrinsic formulation of this constraint on the LG estimator $\widehat{\mathbf{X}}_0$, for standard estimation, is [10],

$$\mathbf{b}_{\mathbf{Z}|\mathbf{X}} = \boldsymbol{l}_{G''}(\mathbf{X}_0, \mathbf{X}), \ \forall \mathbf{X} \in G. \tag{12}$$

In a similar fashion as the BB in the Euclidean space:

Definition 3.1 (LG-BB on LG): The LG-BB, denoted $\mathbf{P}_{\text{LG-BB}}$, is defined as the minimum value of the intrinsic MSE (11) under the intrinsic uniform unbiasedness constraint (12),

$$\mathbf{P}_{\text{LG-BB}} = \min_{\widehat{\mathbf{H}(\mathbf{X}_0)}} \mathbf{C}_{\mathbf{Z}|\mathbf{X}_0} \quad \text{s.t. } \mathbf{b}_{\mathbf{Z}|\mathbf{X}} = \boldsymbol{l}_{G''}(\mathbf{X}_0, \mathbf{X}), \forall\, \mathbf{X} \in G. \tag{13}$$

A lower bound $\mathbf{P}$ on the LG-MSE is then derived from a discretization of the constraint (12) on a set of test points $\mathbf{X}^{(1:L)} = \{\mathbf{X}^{(1)}, \ldots, \mathbf{X}^{(L)}\} \in G$ yielding the inequality,

$$\mathbf{C}_{\mathbf{Z}|\mathbf{X}_0} \succeq \mathbf{P}, \ \mathbf{P} = \boldsymbol{\Delta}_G\, \mathbf{R}^{-1}_{\mathbf{v}\mathbf{x}_0}\boldsymbol{\Delta}_G^\top. \tag{14}$$

where $\succeq$ means that $\mathbf{C}_{\mathbf{Z}|\mathbf{X}_0}(\mathbf{X}_0, \widehat{\mathbf{X}_0}) - \mathbf{P}$ is positive semi-definite (Löwner ordering [16]), and

$$\boldsymbol{\Delta}_G^\top \stackrel{def}{=} \begin{bmatrix} \boldsymbol{l}_{G''}\left(\mathbf{X}_0, \mathbf{X}^{(1)}\right)^\top \\ \vdots \\ \boldsymbol{l}_{G''}\left(\mathbf{X}_0, \mathbf{X}^{(L)})\right)^\top \end{bmatrix}, \tag{15}$$

$$\mathbf{R}_{\mathbf{v}_{\mathbf{X}_0}} = \mathbb{E}_{p(\mathbf{Z}|\mathbf{X}_0)}\left(\mathbf{v}_{\mathbf{X}_0}\left(\mathbf{Z}|\mathbf{X}^{(1:L)}\right) \mathbf{v}_{\mathbf{X}_0}\left(\mathbf{Z}|\mathbf{X}^{(1:L)}\right)^\top\right).$$

with $\mathbf{v}_{\mathbf{X}_0}\left(\mathbf{Z}|\mathbf{X}^{(1:L)}\right) = \left[v_{\mathbf{X}_0}\left(\mathbf{Z}|\mathbf{X}^{(1)}\right), \ldots, v_{\mathbf{X}_0}\left(\mathbf{Z}|\mathbf{X}^{(L)}\right)\right]^\top$ is the vector gathering the likelihood ratios $v_{\mathbf{X}_0}\left(\mathbf{Z}|\mathbf{X}^{(l)}\right) = \dfrac{p(\mathbf{Z}|\mathbf{X}^{(l)})}{p(\mathbf{Z}|\mathbf{X}_0)}$, $\forall l \in \{1, \ldots, L\}$.

Definition 3.2 (LG-CRB): The inequality (14) is the cornerstone for deriving the LG-CRB; selecting the test points

$$\mathbf{X}^{(1:L)} = \{\mathbf{X}_0, \mathbf{X}_0 \operatorname{Exp}_G^\wedge(\mathbf{i}_1\,\delta_1), .., \mathbf{X}_0 \operatorname{Exp}_G^\wedge(\mathbf{i}_{L-1}\,\delta_{L-1})\},$$

$$\mathbf{i}_l = \Big[\; 0, \ldots, \underbrace{1}_{l\text{th component}} \ldots, 0 \;\Big]^\top \in \mathbb{R}^m. \tag{16}$$

yields the definition of the LG-CRB, when δ_l tends to 0,

$$\mathbf{P}_{\text{LG-CRB}} = \mathbb{E}_{p(\mathbf{Z}|\mathbf{X}_0)}\left(\mathbf{s}(\mathbf{Z}|\mathbf{X}_0)\mathbf{s}(\mathbf{Z}|\mathbf{X}_0)^\top\right)^{-1} \tag{17}$$

where $\mathbf{s}(\mathbf{Z}|\mathbf{X}_0) = \left.\frac{\partial \log p\left(\mathbf{Z}|\mathbf{X}_0 \operatorname{Exp}_G^\wedge(\boldsymbol{\delta})\right)}{\partial \boldsymbol{\delta}}\right|_{\boldsymbol{\delta}=\mathbf{0}}$.

Important Remark: In the particular case of unimodular LGs [17] (such as $SO(n)$ and $SE(n)$), provided that the function $\boldsymbol{\delta} \to \log p\left(\mathbf{Z}, \mathbf{Y}|\mathbf{X}_0 \operatorname{Exp}_G^\wedge(\boldsymbol{\delta})\right)$ is sufficiently regular, the aforementioned expression (17) can be further simplified,

$$\mathbb{E}_{p(\mathbf{Z}|\mathbf{X}_0)}\left(\mathbf{s}(\mathbf{Z}|\mathbf{X}_0)\, \boldsymbol{s}(\mathbf{Z}|\mathbf{X}_0)^\top\right) = -\mathbb{E}_{p(\mathbf{Z}|\mathbf{X}_0)} \left(\left.\frac{\partial^2 \log p(\mathbf{Z}|\mathbf{X}_0 \operatorname{Exp}_G^\wedge(\boldsymbol{\delta}_1) \operatorname{Exp}_G^\wedge(\boldsymbol{\delta}_2))}{\partial\, \boldsymbol{\delta}_1\, \partial\, \boldsymbol{\delta}_2}\right|_{\boldsymbol{\delta}_1, \boldsymbol{\delta}_2 = \mathbf{0}}\right). \tag{18}$$

B. LG-CRB for Euclidean observations with Unknown Covariance Matrix

Let us consider the following Euclidean model:

$$p(\mathbf{z}_i|\mathbf{X}_0, \boldsymbol{\Sigma}) = \mathcal{N}(\mathbf{z}_i; \mathbf{f}_i(\mathbf{X}_0), \boldsymbol{\Sigma}), \tag{19}$$

where $\mathbf{f}_i : G \to \mathbb{R}^s$ is a smooth function, and the covariance matrix $\boldsymbol{\Sigma} \in \mathbb{R}^{s \times s}$ is also unknown.

*Definition 3.3 (**C-LG-CRB for Gaussian Euclidean observations**):* For the observations $\mathbf{z} = \{\mathbf{z}_1, \ldots, \mathbf{z}_N\}$, the instrinsic CRB of the parameters $(\mathbf{X}_0, \boldsymbol{\Sigma})$ is lower bounded with $\mathcal{I}_{G'}$ wherein,

$$\mathcal{I}_{G'} = \begin{bmatrix} \mathbf{A} & \mathbf{0}_{d \times g} \\ \mathbf{0}_{g \times d} & \mathbf{B} \end{bmatrix}, \tag{20}$$

where

$$\mathbf{A} = \sum_{i=1}^{N} \mathcal{L}^{R}_{\mathbf{f}_i(\mathbf{X}_0)}{}^\top \boldsymbol{\Sigma}^{-1} \mathcal{L}^{R}_{\mathbf{f}_i(\mathbf{X}_0)} \tag{21}$$

$$\mathcal{L}^{R}_{\mathbf{f}_i(\mathbf{X}_0)} = \left.\frac{\partial \mathbf{f}_i(\mathbf{X}_0 \operatorname{Exp}_G^\wedge(\boldsymbol{\delta}))}{\partial \boldsymbol{\delta}}\right|_{\boldsymbol{\delta}=\mathbf{0}} \tag{22}$$

$$\mathbf{B} = \frac{N}{2} \operatorname{diag}\Big[\underbrace{1, \ldots, 1}_{s}, \underbrace{2, \ldots, 2}_{\frac{s\,(s-1)}{2}}\Big]. \tag{23}$$

C. LG-CRB for LG-Gaussian observations and Slepian-Bangs formula

The success of the Euclidean CRB is largely attributed to the Slepian-Bangs formula [18], [19], which provides a closed-form expression for the Fisher information free of expectation operators. This formula is particularly significant in the fundamental scenario where the observations $\mathbf{z} \in \mathbb{R}^N$ are modeled as Gaussian, with a mean vector $\boldsymbol{\mu}(\mathbf{x}_0)$ and a covariance matrix $\boldsymbol{\Sigma}(\mathbf{x}_0)$, both parameterized by the unknown parameter vector $\mathbf{x}_0 \in \mathbb{R}^P$. In this section, we present the full Slepian-Bangs (F-LG-SP). For further details regarding the derivation of these formulae, readers can refer to [20]. We consider now the model following a Concentrated Gaussian Distribution (CGD):

$$\mathbf{Z}_i = \mathbf{F}_i(\mathbf{X}_0) \operatorname{Exp}_{G'}^\wedge(\boldsymbol{\epsilon}_i), \quad \boldsymbol{\epsilon}_i \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}(\mathbf{X}_0)). \tag{24}$$

where $\mathbf{F}_i : G \to G'$ is a smooth function.

*Theorem 1 (**LG-F-SP for a CGD**): the LG-F-SP $\mathcal{I}$ on $\mathbf{X}_0$ for the observation model (24) is given by:*

$$\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_2^\top + \mathcal{I}_3 \tag{25}$$

wherein,

$$\mathcal{I}_1 = \sum_{i=1}^{N} \mathcal{L}^\top_{\mathbf{F}_i(\mathbf{X}_0)} \boldsymbol{\psi}_i^\top \boldsymbol{\Sigma}(\mathbf{X}_0)^{-1} \boldsymbol{\psi}_i\, \mathcal{L}_{\mathbf{F}_i(\mathbf{X}_0)} \tag{26}$$

$$\mathcal{I}_2 = \frac{1}{2}\mathbb{E}\left(\mathcal{L}^\top_{\mathbf{F}_i(\mathbf{X}_0)} \boldsymbol{\psi}_i^\top\; \boldsymbol{\Sigma}^{-1} \boldsymbol{l}_i\; \{\boldsymbol{l}_i^\top \mathrm{d}\boldsymbol{\Sigma}_1^{-1} \boldsymbol{l}_i, \ldots, \boldsymbol{l}_i^\top \mathrm{d}\boldsymbol{\Sigma}_S^{-1} \boldsymbol{l}_i\}\right) + \frac{1}{2}\mathbb{E}\left(\mathcal{L}^\top_{\mathbf{F}_i(\mathbf{X}_0)} \boldsymbol{\psi}_i^\top\; \boldsymbol{\Sigma}^{-1} \boldsymbol{l}_i\right) \mathrm{dlog}|\boldsymbol{\Sigma}|^\top \tag{27}$$

$$(\mathcal{I}_3)_{k,l} = \frac{N}{2} \operatorname{tr}\left(\boldsymbol{\Sigma}^{-1} \mathrm{d}\boldsymbol{\Sigma}_k\, \boldsymbol{\Sigma}^{-1} \mathrm{d}\boldsymbol{\Sigma}_l\right) \forall (k, l) \in [\![1, \ldots, S]\!]^2 \tag{28}$$

where we make use of the following notations:

$$\begin{aligned} \mathcal{L}_{\mathbf{F}_i(\mathbf{X}_0)} &= \left.\frac{\partial\, \boldsymbol{l}_{G'}(\mathbf{F}_i(\mathbf{X}_0 \operatorname{Exp}_G^\wedge(\boldsymbol{\delta})), \mathbf{Z}_i)}{\partial \boldsymbol{\delta}}\right|_{\boldsymbol{\delta}=\mathbf{0}} \in \mathbb{R}^{S' \times S}, \\ \boldsymbol{l}_i &= \boldsymbol{l}_{G'}(\mathbf{F}_i(\mathbf{X}_0), \mathbf{Z}_i) \in \mathbb{R}^{S'} \\ \boldsymbol{\psi}_i &= \boldsymbol{\psi}_{G'}(\boldsymbol{l}_i) \in \mathbb{R}^{S' \times S'}, \\ \boldsymbol{\Sigma} &= \boldsymbol{\Sigma}(\mathbf{X}_0) \in \mathbb{R}^{S' \times S'}, \\ \mathrm{d}\boldsymbol{\Sigma}_l &= \left.\frac{\partial \boldsymbol{\Sigma}(\mathbf{X}_0 \operatorname{Exp}_G^\wedge(\boldsymbol{\delta}))}{\partial \delta_l}\right|_{\boldsymbol{\delta}=\mathbf{0}} \in \mathbb{R}^{S' \times S'} \forall l \in [\![1, \ldots, S]\!] \end{aligned} \tag{29}$$

and $\mathrm{dlog}|\boldsymbol{\Sigma}| = \left.\dfrac{\partial \log|\boldsymbol{\Sigma}(\mathbf{X}_0 \operatorname{Exp}_G^\wedge(\boldsymbol{\delta}))|}{\partial \boldsymbol{\delta}}\right|_{\boldsymbol{\delta}=\mathbf{0}} \in \mathbb{R}^S$.

D. Closed-form expression on $SO(3)$

We derive a closed-form expression for the Fisher information matrix based on the model assumed in the subsection III-C, based on the well- known Wabha's problem [21]. It consists in finding the unknown rotation $\mathbf{X}_0 \in SO(3)$ connecting two 3D point clouds $\{\mathbf{z}_i\}_{i=1}^N$ and $\{\mathbf{p}_i\}_{i=1}^N$, expressed in two different frames. This can be modeled as,

$$\mathbf{z}_i = \mathbf{X}_0\, \mathbf{p}_i + \mathbf{n}_i \quad \forall i \in \{1, \ldots, N\} \;\; \mathbf{n}_i \sim \mathcal{N}(\mathbf{0}, \mathbf{Q}). \tag{30}$$

In addition to the measurement noise of $\mathbf{z}_i$, the points $\{\mathbf{p}_i\}_{i=1}^N$ are also measured with some uncertainties. They can be modeled by the following Gaussian distribution with mean $\mathbf{p}_i^p$ and covariance matrix $\mathbf{Q}^p$,

$$p(\mathbf{p}_i) = \mathcal{N}(\mathbf{p}_i; \mathbf{p}_i^p, \mathbf{Q}^p). \tag{31}$$

Consequently, the distribution of $\mathbf{z}_i$ knowing $\mathbf{X}_0$ can be rewritten by using the conditional property and Gaussian distribution properties,

$$p(\mathbf{z}_i|\mathbf{X}_0) = \mathcal{N}(\mathbf{z}_i; \mathbf{X}_0\mathbf{p}_i^p, \mathbf{X}_0\,\mathbf{Q}^p\,\mathbf{X}_0^\top + \mathbf{Q}). \tag{32}$$

Consequently, the previous model can be reformulated on the LG $G' = \mathbb{R}^3$ using the compact CGD form:

$$\mathbf{Z}_i = \mathbf{F}_i(\mathbf{X}_0)\,\mathrm{Exp}^\wedge_{\mathbb{R}^3}(\boldsymbol{\epsilon}_i)\,, \qquad \boldsymbol{\epsilon}_i \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}(\mathbf{X}_0)), \tag{33}$$

with $\mathbf{H}_i(\mathbf{X}_0) \triangleq \begin{bmatrix} \mathbf{I} & \mathbf{X}_0\mathbf{p}_i^p \\ \mathbf{0} & 1 \end{bmatrix}$, $\mathrm{Exp}^\wedge_{\mathbb{R}^3}(\boldsymbol{\epsilon}_i) \triangleq \begin{bmatrix} \mathbf{I} & \boldsymbol{\epsilon}_i \\ \mathbf{0} & 1 \end{bmatrix}$ and $\boldsymbol{\Sigma}(\mathbf{X}_0) \triangleq \mathbf{X}_0\,\mathbf{Q}^p\,\mathbf{X}_0^\top + \mathbf{Q}$. We consider the LG observation defined by equation (33). Furthermore, let us define $\{\mathbf{G}_l\}_{l=1}^3$ a basis of $\mathfrak{se}(3)$. The Slepian-Bangs formula is given by $\forall (k,l) \in \{1,2,3\}^2$:

$$\mathcal{I} = \mathcal{I}_R + \mathcal{I}_\Sigma \tag{34}$$

$$(\mathcal{I}_R)_{k,l} = \sum_{i=1}^{N} (\mathbf{p}_i^p)^\top\,\mathbf{G}_k^\top\,\mathbf{X}_0^\top \boldsymbol{\Sigma}(\mathbf{X}_0)^{-1}\mathbf{X}_0\,\mathbf{G}_l\,\mathbf{p}_i^p \tag{35}$$

$$(\mathcal{I}_\Sigma)_{k,l} = \frac{N}{2}\,\mathrm{tr}\left(\boldsymbol{\Sigma}(\mathbf{X}_0)^{-1}\,\left(\mathbf{X}_0\,\mathbf{G}_k\,\mathbf{Q}^p\,\mathbf{X}_0^\top + \mathbf{X}_0\,\mathbf{Q}^p\,\mathbf{G}_k^\top\,\mathbf{X}_0^\top\right)\right.$$
$$\left.\boldsymbol{\Sigma}(\mathbf{X}_0)^{-1}\,\left(\mathbf{X}_0\,\mathbf{G}_l\,\mathbf{Q}^p\,\mathbf{X}_0^\top + \mathbf{X}_0\,\mathbf{Q}^p\,\mathbf{G}_l^\top\,\mathbf{X}_0^\top\right)\right) \tag{36}$$

IV. Bayesian Cramér-Rao bound on LGs

This section focuses on the Bayesian Cramér-Rao bound on LGs, referred to as LG-BCRB.

A. Bayesian estimation problem

We address the following scenario: a matrix of unknown parameters, denoted as $\mathbf{X}_0$, is defined on a unimodular Lie group (LG) G and is estimated from a set of observations $\mathbf{Z} = \{\mathbf{Z}_1, \ldots, \mathbf{Z}_n\}$, where each $\mathbf{Z}_i$ (for $i = 1, \ldots, n$) also belongs to a unimodular LG G'. The set $\mathbf{Z}$ takes values in the product LG ${G'}^n$ and we assume that $\mathbf{X}_0$ is *a priori* distributed according to a probability density function (pdf) $p(\mathbf{X}_0)$ and is related to $\mathbf{Z}$ through the likelihood $p(\mathbf{Z}|\mathbf{X}_0)$. Additionally, $\widehat{\mathbf{X}_0}$ represents a posterior estimator of $\mathbf{X}_0$, derived from $p(\mathbf{X}_0|\mathbf{Z})$. The Bayesian IMSE is defined by:

$$\mathbb{E}\left(\|\mathrm{Log}^\vee_G\left(\mathbf{X}_0^{-1}\,\widehat{\mathbf{X}_0}\right)\|^2\right) =$$
$$\iint \|\mathrm{Log}^\vee_G\left(\mathbf{X}_0^{-1}\,\widehat{\mathbf{X}_0}\right)\|^2 p(\mathbf{Z},\mathbf{X}_0)\lambda_S(d\mathbf{Z}, d\mathbf{X}_0)$$
$$\triangleq \mathbf{MSE}_{bay}(\widehat{\mathbf{X}_0}, \mathbf{X}_0) \tag{37}$$

B. Inequality on the LG-IMSE

Theorem 2: The correlation matrix of the estimation error satisfies:

$$\mathbf{MSE}_{bay}(\mathbf{X}_0, \widehat{\mathbf{X}_0}) \succeq$$
$$\mathbb{E}\left(\psi_G\left(\mathrm{Log}^\vee_G\left(\mathbf{X}_0^{-1}\,\widehat{\mathbf{X}_0}\right)\right)\right)\mathbf{P}_{bay}\mathbb{E}\left(\psi_G\left(\mathrm{Log}^\vee_G\left(\mathbf{X}_0^{-1}\,\widehat{\mathbf{X}_0}\right)\right)^\top\right) \tag{38}$$

where $\psi_G(.)$ is defined by equation (8), and $\mathbf{P}_{bay} = \mathcal{I}^{-1}$, with $\mathcal{I}$ representing the expected value of the Hessian matrix on the Lie group (LG) of $-\log p(\mathbf{Z},\mathbf{X}_0)$. This matrix is referred to as the *LG-Bayesian information matrix* and is expressed as follows:

$$\mathcal{I} = -\mathbb{E}\left(\frac{\partial^2 \log p\left(\mathbf{Z}, \mathbf{X}_0\,\mathrm{Exp}^\wedge_G(\boldsymbol{\epsilon}_1)\,\mathrm{Exp}^\wedge_G(\boldsymbol{\epsilon}_2)\right)}{\partial\boldsymbol{\epsilon}_1\partial\boldsymbol{\epsilon}_2}\bigg|_{\boldsymbol{\epsilon}_1,\boldsymbol{\epsilon}_2=\mathbf{0}}\right). \tag{39}$$

C. Expression of the LG-BCRB

The inequality (38) does not directly provide a lower bound. This is due to two main factors: first, the intrinsic mean squared error appears implicitly on the right-hand side of (38); second, the left Jacobian of G, denoted as $\psi_G(.)$, lacks a tractable expression. To address this, we first develop the left Jacobian according to equation (8). Assuming that $p(\mathbf{X}_0|\mathbf{Z})$ is not overly dispersed, we can approximate the left Jacobian of G at first order, given that non-negligible probability matrices $\mathbf{X}_0$ are close to $\widehat{\mathbf{X}_0}$ and (38) yields:

$$\mathbb{E}\left(\mathrm{Log}^\vee_G\left(\mathbf{X}_0^{-1}\,\widehat{\mathbf{X}_0}\right)\,\mathrm{Log}^\vee_G\left(\mathbf{X}_0^{-1}\,\widehat{\mathbf{X}_0}\right)^\top\right) \succeq$$
$$\mathbf{P} - \frac{1}{2}\,\mathbf{A} + O\left(\|\mathrm{Log}^\vee_G(\mathbf{X}_0^{-1}\widehat{\mathbf{X}_0})\|^2\right), \tag{40}$$

where we define $\mathbf{E} = \mathbb{E}\left(ad_G(\mathrm{Log}^\vee_G(\mathbf{X}_0^{-1}\,\widehat{\mathbf{X}_0}))\right)$ and

$$\mathbf{A} = \mathbf{P}\,\mathbf{E}^\top + \mathbf{E}\,\mathbf{P}. \tag{41}$$

Taking the trace of (40) provides an inequality for $\mathbb{E}\left[\|\boldsymbol{\delta}_{\widehat{\mathbf{X}_0}}\|^2\right]$, which corresponds to the Bayesian LG-MSE. This allows us to derive analytic formulas for the proposed LG-BCRB for the Lie groups of interest, $SO(3)$ and $SE(3)$. For simplicity, we will omit the curvature terms highlighted in (40), as they contribute negligibly.

Theorem 3 (LG-BCRB on $SE(3)$): The proposed LG-BCRB on $SE(3)$ is expressed as:

$$\text{LG-BCRB} = \left(-\frac{\sqrt{2}\,\alpha}{2} + \sqrt{\frac{\alpha^2}{2} + \mathrm{tr}\left[\mathbf{P}\right]}\right)^2. \tag{42}$$

where $\alpha = \mathrm{tr}\left[\mathbf{P}_3\,\mathbf{P}_3^\top\right]$ with $\mathbf{P}_3$ a sub-matrix resulting from the decomposition of $\mathbf{P}$ in four blocks $\in \mathbb{R}^{3\times 3}$. More precisely, we define:

$$\mathbf{P} = \left[\begin{array}{c|c} \mathbf{P}_1 & \mathbf{P}_2 \\ \hline \mathbf{P}_3 & \mathbf{P}_4 \end{array}\right], \tag{43}$$

where $\mathbf{P}_3 = \mathbf{P}_2^\top$ while $\mathbf{P}_1$ and $\mathbf{P}_4$ are symmetric.

D. Closed-form for LG-Gaussian on $SE(3)$

We formulate the LG-BCRB for an inference problem on $SE(3)$, where both the likelihood and the prior distribution are characterized as CGDs $\in SE(3)$. Assuming we have a set of observations $\mathbf{Z} = \{\mathbf{Z}_1, \ldots, \mathbf{Z}_n\}$ that take values in $SE(3)$, these observations are mutually independent and follow the model

$$\mathbf{Z}_i = \mathbf{X}_0 \operatorname{Exp}^{\wedge}_{SE(3)}(\mathbf{e}_i) \quad \forall i \in \{1, \ldots, n\}, \tag{44}$$

where $\mathbf{e}_i \sim \mathcal{N}_{\mathbb{R}^6}(\mathbf{0}, \mathbf{S})$. In a Bayesian context, we treat $\mathbf{X}_0$ as a random variable, with prior information represented by a distribution defined as:

$$\mathbf{X}_0 = \mathbf{X}_p \operatorname{Exp}^{\wedge}_{SE(3)}(\mathbf{e}_M) \quad \text{with} \quad \mathbf{e}_M \sim \mathcal{N}_{\mathbb{R}^6}(\mathbf{0}, \mathbf{\Sigma}_M), \tag{45}$$

where $\mathbf{X}_p \in SE(3)$. The LG-BCRB depends on the LG-Bayesian information matrix (39), which in turn relies on the Hessian matrix of the logarithm of the joint distribution $p(\mathbf{X}_0, \mathbf{Z})$, denoted $\mathbf{H}(\mathbf{X}_0)$. Using Bayes' rule, we can decompose $\mathbf{P}$ into two components:

$$\mathbf{P} = -\left(\mathbb{E}\left(\mathbf{H}_{\mathbf{Z}}(\mathbf{X}_0)\right) + \mathbb{E}\left(\mathbf{H}_{\mathbf{X}}(\mathbf{X}_0)\right)\right)^{-1}. \tag{46}$$

where $\mathbf{H}_{\mathbf{Z}}(\mathbf{X}_0)$ and $\mathbf{H}_{\mathbf{X}_0}(\mathbf{X}_0)$ represent the Hessian matrices of $\log p(\mathbf{Z}|\mathbf{X}_0)$ and $\log p(\mathbf{X}_0)$ with respect to $\mathbf{X}_0$, respectively. By using the properties of the LG-Gaussian, $\mathbb{E}\left(\mathbf{H}_{\mathbf{Z}}(\mathbf{X}_0)\right)$ and $\mathbb{E}\left(\mathbf{H}_{\mathbf{Z}}(\mathbf{X}_0)\right)$ depends on $\psi_G(.)$ and perform an approximation at order 1 of $\psi_G(.)$ Then $\mathbf{P}$ can be expressed as:

$$\mathbf{P} \simeq \left[n\,\mathbf{S}^{-1} + \frac{n}{4}\sum_{j=1}^{6}\sum_{k=1}^{6}(\mathbf{A}_Z)_{j,k} + \mathbf{\Sigma}_M^{-1} - \frac{1}{2}\sum_{j=1}^{6}\sum_{k=1}^{6}\left(\mathbf{A}_{M1} + \mathbf{A}_{M2} - \frac{1}{2}\mathbf{A}_{M3}\right)_{j,k} \right]^{-1}. \tag{47}$$

and $\mathbf{A}_{M1}$, $\mathbf{A}_{M2}$, and $\mathbf{A}_{M3}$ are tensors of size $6 \times 6 \times 6 \times 6$ defined as follows:

$$\left(\mathbf{A}_{M1}\right)_{j,k} = (\mathbf{\Sigma}_M)_{j,k}\, ad_{SE(3)}(\mathbf{c}_j)^\top\, ad_{SE(3)}(\mathbf{c}_k)^\top\, \mathbf{\Sigma}_M^{-1} \tag{48}$$

$$\left(\mathbf{A}_{M2}\right)_{j,k} = (\mathbf{\Sigma}_M)_{j,k}\, \mathbf{\Sigma}_M^{-1}\, ad_{SE(3)}(\mathbf{c}_j)\, ad_{SE(3)}(\mathbf{c}_k) \tag{49}$$

$$\left(\mathbf{A}_{M3}\right)_{j,k} = (\mathbf{\Sigma}_M)_{j,k}\, ad_{SE(3)}(\mathbf{c}_j)^\top\, \mathbf{\Sigma}_M^{-1}\, ad_{SE(3)}(\mathbf{c}_k). \tag{50}$$

The LG-BCRB can then be calculated according to (42). Notably, this bound effectively captures significant terms related to the geometric structure of $SE(3)$ through the tensors $\mathbf{A}_Z$, $\mathbf{A}_{M1}$, $\mathbf{A}_{M2}$, and $\mathbf{A}_{M3}$.

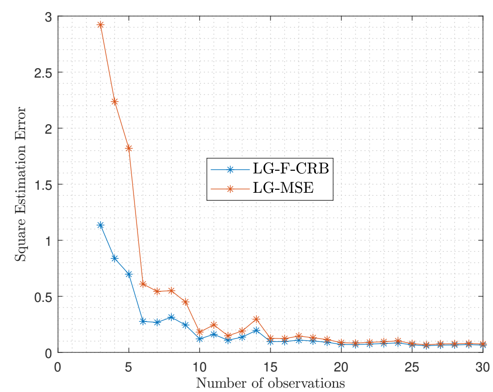
V. Simulation Results

In this section, we demonstrate the applicability of the developed bounds through two representative examples. First, we validate the deterministic LG-Cramér-Rao bound (LG-CRB) using the model presented in Subsection (III-C) on $SO(3)$. Second, we implement and validate the Bayesian LG-Bayes Cramér-Rao bound (LG-BCRB) derived in Section (IV-D), utilizing the models described in Equations (44) and (45).

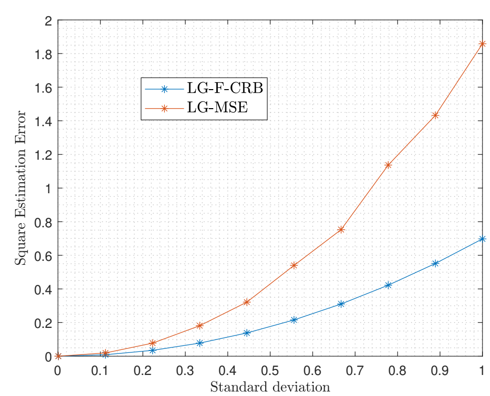
A. LG-CRB on SO(3)

In this part, we propose to test and validate the LG-CRB formula for the complete Slepian-Bangs formula. To achieve this, we first simulate observations using the formula with arbitrary values of $\mathbf{p}_i^p$, $\mathbf{Q} = \sigma^2\mathbf{I}_3$, and $\mathbf{Q}^p = \sigma_p^2\mathbf{I}_3$.

Next, we compare the inverse of the LG-F-SP (34), which yields the LG-F-CRB, with the empirical LG-MSE given by $\frac{1}{N_r}\sum_{nr=1}^{N_r}\|\operatorname{Log}_G^{\vee}\left(\mathbf{X}_0^{-1}\widehat{\mathbf{X}_0}^{(nr)}\right)\|^2$ where $\widehat{\mathbf{X}_0}^{(nr)}$ is the n_r-th realization of the maximum likelihood estimator of the model (30), minimizing:$\sum_{i=1}^{N}\|\boldsymbol{z}_i - \mathbf{X}_0\boldsymbol{p}_i^p\|^2_{\mathbf{\Sigma}^{-1}(\mathbf{X}_0} + N\log|\mathbf{\Sigma}(\mathbf{X}_0)|$.This is obtained iteratively using a Gauss-Newton algorithm. In figures 1a and 1b, we plot both LG-MSE and LG-CRB with respect to the number of observations for $\sigma^2 = 0.01^2$ and $\sigma_p^2 = 1^2$, and with respect to varying values of the standard deviation σ for a fixed $N = 5$.



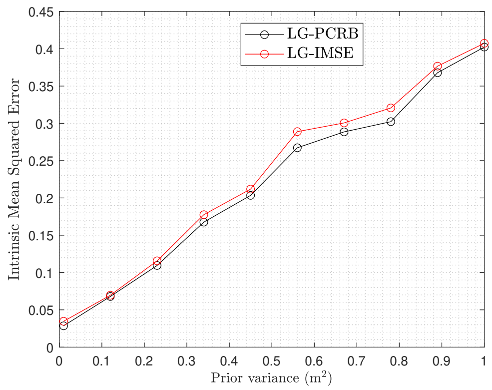
(a) LG-CRB and LG-MSE w.r.t the number of observations N.



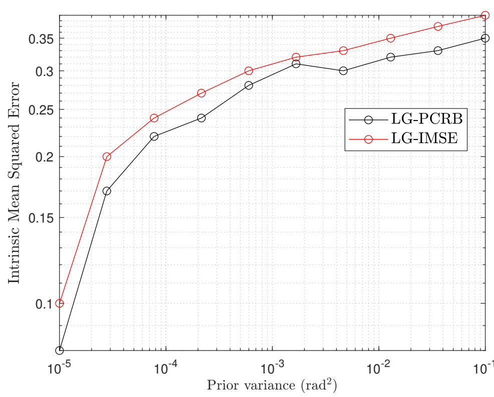
(b) LG-CRB and LG-MSE w.r.t varying values of the measurement noise σ^2.

We observe the consistency of the LG-CRB with respect to the LG-MSE, particularly in its asymptotic behavior. Specifically, in Fig. 1a, the LG-MSE aligns with the LG-F-CRB as the number of observations increases. Furthermore, in Fig. 1b, the LG-F-CRB and the LG-MSE align when the measurement noise variance is low. As the noise variance increases, the LG-MSE deviates due to bias. This behavior, consistent with the Euclidean case, validates the LG-CRB.

B. LG-BCRB on SE(3)



(a) Influence of the variance of the translation components of $\mathbf{\Sigma}_M$



(b) Influence of the variance of the rotation components of $\mathbf{\Sigma}_M$

Now, we propose to validate the LG-BCRB. To do that, we address a well-known problem in computer vision: estimating the pose of a camera, defined by a matrix in SE(3). We assume the availability of a single measurement $\mathbf{Z} \in SE(3)$ provided by a Point-n-Perspective (PnP) module [47]. This method geometrically computes the pose with some uncertainty, allowing the PnP observation to be modeled on the Lie group $SE(3)$ as (47) where $\mathbf{X}_0 \in SE(3)$ represents the true camera pose, and $\text{Exp}^{\wedge}_{SE(3)}(\boldsymbol{\epsilon}_i)$ denotes the error on $SE(3)$ due to the PnP inference. Additionally, we assume that the practitioner has prior information about the actual position and orientation of the camera, which can be modeled as (45). To validate the bound, we examine its behavior as the *a priori* distributions of the position and orientation components of $\mathbf{\Sigma}_M$ evolve by assuming assume isotropic variances for these components. As expected, when the *a priori* dispersion of the position components increases significantly, the estimation error tends to worsen. Figure 2a shows that the bound adjusts accordingly, following the same trend as the error but remaining slightly lower. Similarly, the LG-IMSE increases when the variance of the orientation components becomes large (approaching $10^{-1}, \text{rad}^2$). This trend is also reflected in the bound, as seen in figure 2b.

VI. Conclusion

This work presents recent advancements in the development of Cramér-Rao bounds for Lie groups (LGs). Specifically, we have reviewed their formulation within both deterministic and Bayesian frameworks and established their consistency through numerical validation on the LGs $SO(3)$ and $SE(3)$. The findings open several significant avenues for future research. One particularly promising direction is the extension of these results to encompass LG parameters jointly with parameters from other manifolds. Another key perspective involves the development of recursive computation schemes for Markovian dynamic systems defined on LGs.

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